ON RUMER'S INVARIANT THEORY OF GRAVITATIONAL WAVES

K. D. KRORI

DEPARTMENT OF PHYSICS, COTTON COLLEGE, GAUHATI (ASSAM)

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ABSTRACT. Propagation of gravitational potential and gravitational field-strength has been considered here on the basis of Rumer's Invariant Theory of Gravitational Waves. Finally some special cases of weak field have been discussed.

INTRODUCTION

Rumer (1962) has constructed a theory of gravitation by breaking up the fourth-rank Riemann tensor $R_{\mu\nu\alpha\beta}$ into a sum of two 10-component tensors $M_{\mu\nu\alpha\beta}$ and $F_{\mu\nu\alpha\beta}$, both of which have the symmetry of Riemann tensor (anti-symmetrical in μ and ν as well as in α and β) :

$$R_{\mu\nu\alpha\beta} = M_{\mu\nu\alpha\beta}+F_{\mu\nu\alpha\beta} \qquad \dots \quad (1)$$

$M_{\mu\nu\alpha\beta}$ is called the matter-tensor and $F_{\mu\nu\alpha\beta}$ the gravitational field-strength tensor. The expression for $F_{\mu\nu}{}^{\alpha\beta}$ can be written as

$$\begin{aligned} F_{\mu\nu}{}^{\alpha\beta} &= R_{\mu\nu}{}^{\alpha\beta}-M_{\mu\nu}{}^{\alpha\beta} \\ &= R_{\mu\nu}{}^{\alpha\beta}+\tfrac{1}{2}[g_\mu{}^\beta T_\nu{}^\alpha+g_\nu{}^\alpha T_\mu{}^\beta-g_\mu{}^\alpha T_\nu{}^\beta-g_\nu{}^\beta T_\mu{}^\alpha] \\ &\qquad -\tfrac{1}{3}(g_\mu{}^\beta g_\nu{}^\alpha-g_\mu{}^\alpha g_\nu{}^\beta)T \qquad \dots \quad (2) \end{aligned}$$

where $T_\nu{}^\alpha$ is the source-tensor. The field equations can be expressed as

$$(F_{\mu\nu}{}^{\alpha\beta})_\beta = \tfrac{1}{2}[(T_\nu{}^\alpha)_\mu-(T_\mu{}^\alpha)_\nu]-\tfrac{1}{6}(g_\nu{}^\alpha T,_\mu-g^\alpha{}_\mu T,_\nu) \qquad \dots \quad (3)$$

which consists of 20 independent equations. [($)_\beta$ indicates covariant differentiation.]

The purpose of this paper is to discuss the propagation of the gravitational potential $\phi_{\mu\alpha}$ (defined below) and of the gravitational field-strength $F_{\mu\nu\alpha\beta}$ on the basis of Rumer's theory

PROPAGATION OF GRAVITATIONAL POTENTIAL $\phi_{\mu\alpha}$

Considering the anti-symmetry of $F_{\mu\nu\alpha\beta}$ in α and β, it can be written as

$$F_{\mu\nu\alpha\beta} = (\psi_{\mu\nu\alpha})_\beta-(\psi_{\mu\nu\beta})_\alpha \qquad \dots \quad (4)$$

Again, because of anti-symmetry in μ and ν, $\psi_{\mu\nu\alpha}$ can be expressed as

$$\psi_{\mu\nu\alpha} = (\phi_{\mu\alpha})_\nu - (\phi_{\nu\alpha})_\mu \quad \ldots \quad (4a)$$

so that

$$F_{\mu\nu\alpha\beta} = (\phi_{\mu\alpha})_{\nu\beta} - (\phi_{\nu\alpha})_{\mu\beta} - (\phi_{\mu\beta})_{\nu\alpha} + (\phi_{\nu\beta})_{\mu\alpha} \quad \ldots \quad (5)$$

We consider that $\phi_{\mu\alpha}$ is a traceless symmetric tensor of rank 2, obeying the divergence condition :

$$(\phi_\mu{}^\alpha)_\alpha = 0 \quad \ldots \quad (6)$$

Now, from (2) and (5), we have

$$g^{\nu\beta}(\phi_{\mu\alpha})_{\nu\beta} - (\phi_\alpha{}^\beta)_{\mu\beta} - (\phi_\mu{}^\nu)_{\nu\alpha} + (\phi)_{\mu\alpha} = g^{\nu\beta} F_{\mu\nu\alpha\beta}$$

$$= R_{\mu\alpha} - (T_{\mu\alpha} - \tfrac{1}{2} g_{\mu\alpha} T) = 0$$

or (Eddington 1957)

$$g^{\nu\beta}(\phi_{\mu\alpha})_{\nu\beta} - (\phi_\alpha{}^\beta)_{\beta\mu} - g^{\nu\beta} R^\epsilon{}_{\nu\mu\beta}\phi_{\epsilon\alpha} - g^{\nu\beta} R^\epsilon{}_{\alpha\mu\beta}\phi_{\nu\epsilon} - (\phi^\nu{}_\mu)_{\nu\alpha} + (\phi)_{\mu\alpha} = 0 \quad \ldots (7)$$

or

$$\Box \phi_{\mu\alpha} = G^\epsilon{}_\mu \phi_{\epsilon\alpha} + g^{\nu\beta} R^\epsilon{}_{\alpha\mu\beta}\phi_{\nu\epsilon} \quad \ldots \quad (8)$$

remembering the properties attributed to $\phi_{\mu\alpha}$ in (6), (8) gives the propagation of gravitational potential in matter.

PROPAGATION OR GRAVITATIONAL FIELD-STRENGTH $F_{\mu\nu\alpha\beta}$

From (3), we have, by covariant differentiation,

$$\tfrac{1}{2}[(T_{\alpha\nu})_{\mu\delta} - (T_{\alpha\mu})_{\nu\delta}] - \tfrac{1}{6}[g_{\alpha\nu} T,_{\mu\delta} - g_{\alpha\mu} T,_{\nu\delta}] = g^{\gamma\beta}(F_{\mu\nu\alpha\beta})_{\gamma\delta}$$

$$= g^{\gamma\beta}\{(F_{\mu\nu\alpha\beta})_{\delta\gamma} + R^\epsilon{}_{\mu\gamma\delta}F_{\epsilon\nu\alpha\beta} + R^\epsilon{}_{\nu\gamma\delta}F_{\mu\epsilon\alpha\beta} + R^\epsilon{}_{\alpha\gamma\delta}F_{\mu\nu\epsilon\beta} + R^\epsilon{}_{\beta\gamma\delta}F_{\mu\nu\alpha\epsilon}\}$$

$$= g^{\gamma\beta}(F_{\mu\nu\alpha\delta})_{\beta\gamma} + g^{\gamma\beta}(F_{\mu\nu\delta\beta})_{\alpha\gamma} + g^{\gamma\beta}\{R^\epsilon{}_{\mu\gamma\delta}F_{\epsilon\nu\alpha\beta} + R^\epsilon{}_{\nu\gamma\delta}F_{\mu\epsilon\alpha\beta} + R^\epsilon{}_{\alpha\gamma\delta}F_{\mu\nu\epsilon\beta}$$

$$+ R^\epsilon{}_{\beta\gamma\delta}F_{\mu\nu\alpha\epsilon} + (R^\epsilon{}_{\mu\beta\delta}\psi_{\epsilon\nu\alpha} + R^\epsilon{}_{\alpha\beta\delta}\psi_{\mu\nu\epsilon} + R^\epsilon{}_{\nu\beta\delta}\psi_{\mu\epsilon\alpha} - R^\epsilon{}_{\mu\alpha\delta}\psi_{\epsilon\nu\beta} - R^\epsilon{}_{\beta\alpha\delta}\psi_{\mu\nu\epsilon}$$

$$- R^\epsilon{}_{\nu\alpha\delta}\psi_{\mu\epsilon\beta} + R^\epsilon{}_{\mu\alpha\beta}\psi_{\epsilon\nu\delta} + R^\epsilon{}_{\delta\alpha\beta}\psi_{\mu\nu\epsilon} + R^\epsilon{}_{\nu\alpha\beta}\psi_{\mu\epsilon\delta})_\gamma\} \quad \ldots \quad (9)$$

Also, we have

$$\tfrac{1}{2}[(T_{\delta\nu})_{\mu\alpha} - (T_{\delta\mu})_{\nu\alpha}] - \tfrac{1}{6}[g_{\delta\nu}T,_{\mu\alpha} - g_{\delta\mu}T,_{\nu\alpha}] = g^{\gamma\beta}(F_{\mu\nu\delta\beta})_{\gamma\alpha}$$

$$= g^{\gamma\beta}\{(F_{\mu\nu\delta\beta})_{\alpha\gamma} + R^\epsilon{}_{\mu\gamma\alpha}F_{\epsilon\nu\delta\beta} + R^\epsilon{}_{\nu\gamma\alpha}F_{\mu\epsilon\delta\beta} + R^\epsilon{}_{\delta\gamma\alpha}F_{\mu\nu\epsilon\beta} + R^\epsilon{}_{\beta\gamma\alpha}F_{\mu\nu\delta\epsilon}\} \quad \ldots \quad (10)$$

Subtracting (10) from (9)

$$\{\tfrac{1}{2}[(T_{\alpha\nu})_{\mu\delta} - (T_{\alpha\mu})_{\nu\delta}] - \tfrac{1}{6}[g_{\alpha\nu}T,_{\mu\delta} - g_{\alpha\nu}T,_{\nu\delta}]\} - \tfrac{1}{2}[(T_{\delta\nu})_{\mu\alpha} - (T_{\delta\mu})_{\nu\alpha}] - \tfrac{1}{6}[g_{\delta\nu}T,_{\mu\alpha}$$

$$- g_{\delta\mu}T,_{\nu\alpha}]\} = g^{\gamma\beta}(F_{\mu\nu\alpha\beta})_{\gamma\delta} - g^{\gamma\beta}(F_{\mu\nu\delta\beta})_{\gamma\alpha} = \Box F_{\mu\nu\alpha\delta} + g^{\gamma\beta}\{R^\epsilon{}_{\mu\gamma\delta}F_{\epsilon\nu\alpha\beta}$$

$$+ R^\epsilon{}_{\nu\gamma\delta}F_{\mu\epsilon\alpha\beta} + R^\epsilon{}_{\alpha\gamma\delta}F_{\mu\nu\epsilon\beta} + R^\epsilon{}_{\beta\gamma\delta}F_{\mu\nu\alpha\epsilon}) - (R^\epsilon{}_{\mu\gamma\alpha}F_{\epsilon\nu\delta\beta} + R^\epsilon{}_{\nu\gamma\alpha}F_{\mu\epsilon\delta\beta} + R^\epsilon{}_{\delta\gamma\alpha}F_{\mu\nu\epsilon\beta}$$

$$+ R^\epsilon{}_{\beta\gamma\alpha}F_{\mu\nu\delta\epsilon}) + (R^\epsilon{}_{\mu\beta\delta}\psi_{\epsilon\nu\alpha} + R^\epsilon{}_{\alpha\beta\delta}\psi_{\mu\nu\epsilon} + R^\epsilon{}_{\nu\beta\delta}\psi_{\mu\epsilon\alpha} - R^\epsilon{}_{\mu\alpha\delta}\psi_{\epsilon\nu\beta} - R^\epsilon{}_{\beta\alpha\delta}\psi_{\mu\nu\epsilon}$$

$$- R^\epsilon{}_{\nu\alpha\delta}\psi_{\mu\epsilon\beta} + R^\epsilon{}_{\mu\alpha\beta}\psi_{\epsilon\nu\delta} + R^\epsilon{}_{\delta\alpha\beta}\psi_{\mu\nu\delta} + R^\epsilon{}_{\nu\alpha\beta}\psi_{\mu\epsilon\delta})_\gamma\} \quad \ldots \quad (11)$$

writing $\qquad \Box F_{\mu\nu\alpha\delta} = g^{\gamma\beta}(F_{\mu\nu\alpha\delta})_{\beta\gamma}]$

(11) gives the propagation of field-strength in matter.

WEAK FIELD APPROXIMATIONS: SOME SPECIAL CASES

If we consider the field to be weak, then we may write (8) in the following form :

(replacing $g^{\nu\beta}$ by $\delta^{\nu\beta}$)

$$\Box \phi_{\mu\alpha} = G^{\epsilon}{}_{\mu}\phi_{\epsilon\alpha}+\delta^{\nu\beta}M^{\epsilon}{}_{\alpha\mu\beta}\,\phi_{\nu\epsilon} \qquad \ldots \quad (12)$$

Considering a thin distribution of matter or free space. (12) takes the simple form :

$$\Box \phi_{\mu\alpha} = 0 \qquad \ldots \quad (13)$$

This shows that gravitational potential travels with fundamental velocity through a very thin distribution of matter or free space.

Under the weak field condition, (11) comes to (replacing covariant differentiation by ordinary differentiation)

$$\begin{aligned}
&\{\tfrac{1}{2}[T_{\alpha\nu,\mu\delta}-T_{\alpha\mu,\nu\delta}]-\tfrac{1}{6}[\delta_{\alpha\nu}T_{,\mu\delta}-\delta_{\alpha\mu}T_{,\nu\delta}]\}\\
&-\{\tfrac{1}{2}[T_{\delta\nu,\mu\alpha}-T_{\delta\mu,\nu\alpha}]-\tfrac{1}{6}[\delta_{\delta\nu}T_{,\mu\alpha}-\delta_{\delta\mu}T_{,\nu\alpha}]\}\\
&=\Box F_{\mu\nu\alpha\delta}+\delta^{\gamma\beta}\{(M^{\epsilon}{}_{\mu\gamma\delta}F_{\epsilon\nu\alpha\beta}+M^{\epsilon}{}_{\nu\gamma\delta}F_{\mu\epsilon\alpha\beta}+M^{\epsilon}{}_{\alpha\gamma\delta}F_{\mu\nu\epsilon\beta}+M^{\epsilon}{}_{\beta\gamma\delta}F_{\mu\nu\alpha\epsilon})\\
&-(M^{\epsilon}{}_{\mu\gamma\alpha}F_{\epsilon\nu\delta\beta}+M^{\epsilon}{}_{\nu\gamma\alpha}F_{\mu\epsilon\delta\beta}+M^{\epsilon}{}_{\delta\gamma\alpha}F_{\mu\nu\epsilon\beta}+M^{\epsilon}{}_{\beta\gamma\alpha}F_{\mu\nu\delta\epsilon})+(M^{\epsilon}{}_{\mu\beta\delta}\psi_{\epsilon\nu\alpha}\\
&+M^{\epsilon}{}_{\alpha\beta\delta}\psi_{\mu\nu\epsilon}+M^{\epsilon}{}_{\nu\beta\delta}\psi_{\mu\epsilon\alpha}-M^{\epsilon}{}_{\mu\alpha\delta}\psi_{\epsilon\nu\beta}-M^{\epsilon}{}_{\beta\alpha\delta}\psi_{\mu\nu\epsilon}-M^{\epsilon}{}_{\nu\alpha\delta}\psi_{\mu\epsilon\beta}\\
&+M^{\epsilon}{}_{\mu\alpha\beta}\psi_{\epsilon\nu\delta}+M^{\epsilon}{}_{\delta\alpha\beta}\psi_{\mu\nu\epsilon}+M^{\epsilon}{}_{\nu\alpha\beta}\psi_{\mu\epsilon\delta})_{,\gamma}\} \qquad \ldots \quad (14)
\end{aligned}$$

Further simplification can be made by considering vacuum. In that case, we get from (14)

$$\Box F_{\mu\nu\alpha\delta} = 0 \qquad \ldots \quad (15)$$

This means that gravitational field strength also travels with fundamental velocity in vacuum.

PLANE GRAVITATIONAL WAVE: ANOTHER SPECIAL CASE

Following the idea of Weber (1961), we now consider the case of propagation of plane gravitational wave. In such a case the field changes only along one direction in space; we choose for this direction the axis x_1. Now we have from (5)

$$F_{\mu\nu\alpha\beta} = (\phi_{\mu\alpha})_{\nu\beta}-(\phi_{\nu\alpha})_{\mu\beta}-(\phi_{\mu\beta})_{\nu\alpha}+(\phi_{\nu\beta})_{\mu\alpha} \qquad \ldots \quad (5)$$

It is evident for the case under consideration that the derivatives of $\phi_{\mu\alpha}$ and $\phi_{\mu\alpha,\nu}$ with respect to x_2 and x_3 vanish.

The 21 components of $F_{\mu\nu\alpha\beta}$ may be written down as follows from(5) :

$$\left.\begin{array}{lll} F_{2424} = \phi_{22,44} & F_{1224} = -\phi_{22,14} & F_{1212} = \phi_{22,11} \\ F_{2434} = \phi_{23,44} & F_{1234} = -\phi_{23,14} & F_{1213} = \phi_{23,11} \\ F_{3434} = \phi_{33,44} & F_{1324} = -\phi_{23,14} & F_{1313} = \phi_{33,11} \\ & F_{1334} = -\phi_{33,14} & \end{array}\right\} \quad \ldots \quad (16)$$

$$\left.\begin{array}{ll} F_{1223} = 0 & F_{2323} = 0 \\ F_{1323} = 0 & F_{2324} = 0 \\ F_{1423} = 0 & F_{2334} = 0 \end{array}\right\} \quad \ldots \quad (17)$$

$$\left.\begin{array}{ll} F_{1434} = \phi_{13,44} - \phi_{34,14} & F_{1424} = \phi_{12,44} - \phi_{24,14} \\ F_{1314} = \phi_{34,11} - \phi_{13,14} & F_{1214} = \phi_{24,11} - \phi_{12,14} \\ \multicolumn{2}{c}{F_{1414} = \phi_{11,44} - \phi_{14,14} + \phi_{44,11}} \end{array}\right\} \quad \ldots \quad (18)$$

In vacuum, $F_{\mu\nu} = 0$. So, since we are considering weak field, we have from (16), (17) and (18) :

$$\left.\begin{array}{l|l} F_{12} = F_{1424} = 0 & F_{13} = F_{1434} = 0 \\ F_{24} = F_{1214} = 0 & F_{34} = F_{1314} = 0 \\ F_{11} = F_{1414} - F_{1313} - F_{1212} = 0 & F_{14} = F_{1224} + F_{1334} = 0 \\ F_{22} = F_{2424} - F_{1212} = 0 & F_{23} = F_{2434} - F_{1213} = 0 \\ F_{33} = F_{3434} - F_{1313} = 0 & F_{44} = F_{3434} + F_{2424} + F_{1414} = 0 \end{array}\right\} \quad \ldots \quad (19)$$

It is easily seen from (19) that F_{1434}, F_{1314}, F_{1424}, F_{1214} and F_{1414} of (18) are all zero. Then the components of $F_{\mu\nu\alpha\beta}$ that remain are the ten components of (16) which are expressed in terms of only three potential components ϕ_{22}, ϕ_{23} and ϕ_{33}. It is evident from (16), (18) and (19) that

$$\phi_{22,44} + \phi_{33,44} = 0$$

or (on integration), $\quad \ldots \quad (20)$

$$\phi_{22} + \phi_{33} = 0$$

Here we have put the integration constants equal to zero since we are interested in the varying part of the field. Thus, a plane gravitational wave is determined by only two quantities ϕ_{23} and $\phi_{22} = -\phi_{33}$ and is transverse since it is determined by the potential tensor in $x_2 - x_3$ plane only (Landau and Lifshitz, 1962).

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